Cavity approach to the spectral density of non-Hermitian sparse matrices

Tim Rogers and Isaac Pérez Castillo

Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom
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The spectral densities of ensembles of non-Hermitian sparse random matrices are analyzed using the cavity method. We present a set of equations from which the spectral density of a given ensemble can be efficiently and exactly calculated. Within this approach, the generalized Girko's law is recovered easily. We compare our results with direct diagonalisation for a number of random matrix ensembles, finding excellent agreement.

INTRODUCTION

For decades, random matrix theory has been the focus of much attention in both physical and mathematical research, with an ever-expanding and remarkably diverse list of applications (for example, see Ref. [1] for an extensive review of applications in physics). A problem of particular interest is that of determining the spectral density of an ensemble of random matrices. In the early 1950s it was conjectured that the eigenvalues of certain non-Hermitian random matrix ensembles should be spread evenly throughout the unit disk. Now known as Girko's law, this conjecture has been the subject of many rigorous and nonrigorous studies (e.g., Refs. [2–5], and references therein), for various classes of random matrix ensembles.

It is a natural desire, then, to extend our understanding of those ensembles that break away from this law. In the more accessible case of real symmetric matrices, it is known that the introduction of sparsity (that is, many entries of the matrix being zero) results in behavior radically different from that seen in the fully connected limit [6–9]. Sparse real symmetric matrices have been studied extensively, and there exists various approximative schemes [7–9], together with recent exact work [10–12]. As we will see, sparsity also has a significant effect on the spectral density of general non-Hermitian matrices; however, this area has not received the attention it deserves, and consequently a great deal remains unknown.

In this paper, we tackle the problem of computing the spectral density of sparse non-Hermitian matrices using the cavity method [10,13,14]. A simple closed set of equations is uncovered, whose solution characterizes the spectral density of a given matrix. These equations are solved analytically in the fully connected limit, recovering the generalized Girko's law of Ref. [5]. We also present the results of a belief-propagation algorithm to quickly compute the solution for ensembles of large sparse matrices.

PREPARATION

Consider an ensemble $\mathcal{M}$ of $N \times N$ complex, non-Hermitian sparse random matrices. For a given matrix $A \in \mathcal{M}$, we denote the collection of eigenvalues of A by $\{\lambda_i^A : i = 1, \ldots, N\}$; if A is non-Hermitian, it follows that these λ_i^A are complex. For a point $z = x + iy$ in the complex plane, we write the spectral density of A at z as

$$\varrho_A(z, \bar{z}) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \mathrm{Re}\, \lambda_i^A) \delta(y - \mathrm{Im}\, \lambda_i^A). \tag{1}$$

The spectral density of the ensemble, denoted as $\rho(z, \bar{z})$, results from averaging $\varrho_A(z, \bar{z})$ over $\mathcal{M}$. Following, for example, Refs. [5,15,16], we are able to write[1]

$$\varrho_A(z, \bar{z}) = -\frac{1}{\pi N} \lim_{\kappa \to 0} \partial_{\bar{z}} \partial_z \ln \det H, \tag{2}$$

where we have introduced the $2N \times 2N$ matrix

$$H \equiv H(z, \bar{z}; \kappa) = \begin{pmatrix} \kappa \mathbb{1}_N & i(z\mathbb{1}_N - A) \\ i(z\mathbb{1}_N - A)^\dagger & \kappa \mathbb{1}_N \end{pmatrix}. \tag{3}$$

We use $\overline{(\cdots)}$ for the complex conjugate and $(\cdots)^\dagger$ for the conjugate transpose. The next step is to write the determinant of H in terms of a Gaussian integral. In Ref. [5], the replica method was applied to the case of fully connected non-Hermitian Gaussian ensembles, deriving the generalized Girko's law in the limit $N \to \infty$. However, little progress has been made in the study of sparse non-Hermitian matrices. To push forward, we tackle the problem by focusing on the behavior of a large single instance. With a modest amount of foresight, we define N pairs of complex variables

$$\psi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i = 1, \ldots, N \tag{4}$$

and introduce the "Hamiltonian"

$$\mathcal{H}(\boldsymbol{\psi}, z, \bar{z}; \kappa) = \sum_{i=1}^{N} \psi_i^\dagger [\kappa \mathbb{1}_2 + i(x\sigma_x - y\sigma_y)]\psi_i$$
$$- i \sum_{i,j=1}^{N} \psi_i^\dagger (A_{ij}^h \sigma_x - A_{ij}^s \sigma_y)\psi_j, \tag{5}$$

where σ_x and σ_y are the usual Pauli matrices, and we have written $A = A^h + iA^s$, with A^h and A^s Hermitian matrices. Continuing with the statistical mechanics metaphor, we also introduce a "distribution" P and an "average" $\langle \cdots \rangle$:

$$P(\boldsymbol{\psi}) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}(\boldsymbol{\psi}, z, \bar{z}; \kappa)}$$

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[1]We use the notation conventions $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

$$\langle \cdots \rangle = \int \mathcal{D}\boldsymbol{\psi} P(\boldsymbol{\psi})(\cdots), \qquad (6)$$

where $\mathcal{Z}$ is a normalizing constant. Of course, the measure we define here is complex and is manifestly not a real stochastic measure. However, much of the mathematics is unchanged, and it is relatively safe to use this probabilistic analogy. With this groundwork in place, Eq. (2) may finally be written as

$$\varrho_A(z,\bar{z}) = \lim_{\kappa \to 0} \frac{1}{\pi N} \sum_{l=1}^{N} i \partial_{\bar{z}} \langle \psi_l^\dagger \sigma^+ \psi_l \rangle, \qquad (7)$$

where $\sigma^+ = \sigma_x + i\sigma_y$. In this formulation, it becomes clear that the local marginals $P_i(\psi_i)$ will suffice to evaluate ϱ_A. We calculate these marginals using the cavity method.

TREELIKE MATRICES

We consider treelike sparse matrices. Associated with the matrix A there is a weighted, directed graph $\mathcal{G}_A$ on N vertices: a directed edge of weight A_{ij} is drawn from vertex i to vertex j whenever $A_{ij} \neq 0$. We say the graph $\mathcal{G}_A$ (and consequently the matrix A) is "treelike" if short loops are rare. There is a simple link between this graph and the Hamiltonian (5): the interaction of variables ψ_i and ψ_j is encoded in the edges between i and j. We make the following standard definitions: a pair of vertices i, j are neighbors if either A_{ij} or A_{ji} is nonzero; the set of all neighbors of i is denoted ∂i; k_i denotes the number of neighbors of i (the degree of i); the average degree is given by $c = N^{-1} \Sigma_i k_i$.

Notice that, if A is treelike, the variables neighboring ψ_i are correlated mainly through ψ_i. Consider a fictitious situation in which we have removed the variable ψ_i. We are interested in the change to the marginal distributions of the neighboring variables ψ_l with $l \in \partial i$, which we denote by $P_l^{(i)}(\psi_l)$; with their common neighbor now absent, the joint distribution factorizes

$$P^{(i)}(\{\psi_l\}_{l \in \partial i}) = \prod_{l \in \partial i} P_l^{(i)}(\psi_l). \qquad (8)$$

This is known as the Bethe approximation. It is exact on trees and graphs which remain treelike in the limit $N \to \infty$.[2] The cavity marginals $\{P_i^{(j)}\}$ obey simple recursive relations

$$P_i^{(j)}(\psi_i) = \frac{e^{-\mathcal{H}_i}}{Z_i^{(j)}} \int \mathcal{D}(\boldsymbol{\psi}_{\partial i \backslash j}) e^{-\Sigma_{l \in \partial i \backslash j} \mathcal{H}_{il}} \prod_{l \in \partial i \backslash j} P_l^{(i)}(\psi_l), \qquad (9)$$

where $Z_i^{(j)}$ is a normalizing constant, and we have split the Hamiltonian (5) into the contributions from single variables $\mathcal{H}_i$ and from pairs of variables $\mathcal{H}_{ij}$, viz.,

$$\mathcal{H}_i = \psi_i^\dagger [\kappa \mathbb{1}_2 + i(x\sigma_x - y\sigma_y)]\psi_i,$$

$$\mathcal{H}_{ij} = -i\psi_i^\dagger(A_{ij}^h \sigma_x - A_{ij}^s \sigma_y)\psi_j - i\psi_j^\dagger(A_{ji}^h \sigma_x - A_{ji}^s \sigma_y)\psi_i. \qquad (10)$$

If the cavity distributions are known, the real marginal distribution at the vertex i can be recovered by merging the contributions of the neighbors

$$P_i(\psi_i) = \frac{e^{-\mathcal{H}_i}}{Z_i} \int \mathcal{D}(\boldsymbol{\psi}_{\partial i}) e^{-\Sigma_{l \in \partial i} \mathcal{H}_{il}} \prod_{l \in \partial i} P_l^{(i)}(\psi_l). \qquad (11)$$

We see that the set of equations (9) is self-consistently solved by distributions of a bivariate Gaussian type. Specifically, for all $i = 1, \ldots, N$ and all $j \in \partial i$, the distribution $P_i^{(j)}$ has the form

$$P_i^{(j)}(\psi_i) = \frac{1}{Z_i^{(j)}} \exp(-\psi_i^\dagger [\mathcal{C}_i^{(j)}]^{-1} \psi_i), \qquad (12)$$

where $\mathcal{C}_i^{(j)}$ is a 2×2 matrix. Insertion into Eq. (9) yields a set of consistency equations for the matrices $\{\mathcal{C}_i^{(j)}\}$, viz.,

$$\mathcal{C}_i^{(j)} = [F(\mathcal{C}_{\partial i \backslash j}^{(i)}) + \kappa \mathbb{1}_2 + i(x\sigma_x - y\sigma_y)]^{-1} \qquad (13)$$

for all $i = 1, \ldots, N$ and all $j \in \partial i$ and where F is the matrix field

$$F(\mathcal{C}_{\partial i \backslash j}^{(i)}) = \sum_{l \in \partial i \backslash j} (A_{il}^h \sigma_x - A_{il}^s \sigma_y) \mathcal{C}_l^{(i)} (A_{li}^h \sigma_x - A_{li}^s \sigma_y). \qquad (14)$$

Similarly, Eq. (11) gives the "true" covariance matrices

$$\mathcal{C}_i = [F(\mathcal{C}_{\partial i}^{(i)}) + \kappa \mathbb{1}_2 + i(x\sigma_x - y\sigma_y)]^{-1} \qquad (15)$$

for all $i = 1, \ldots, N$. We pause for a moment now to determine the structure of the matrices $\{\mathcal{C}_i^{(j)}\}$. Performing the inverse of H in block form reveals enough information to allow us to write generally

$$\mathcal{C}_i^{(j)} \equiv \begin{pmatrix} a_i^{(j)} & i\bar{b}_i^{(j)} \\ ib_i^{(j)} & d_i^{(j)} \end{pmatrix} \quad \begin{matrix} a_i^{(j)}, d_i^{(j)} \in \mathbb{R}^+ \\ b_i^{(j)} \in \mathbb{C} \end{matrix}. \qquad (16)$$

If one has a solution set to Eq. (13), the "true" local marginals are recovered from Eq. (15). Recalling that the matrices $\{\mathcal{C}_i^{(j)}\}$ and $\{\mathcal{C}_i\}$, are dependent upon z and κ, one may employ Eq. (7) to determine the spectral density in terms of the function $b_i \equiv b_i(z, \bar{z}, \kappa)$,

$$\varrho_A(z,\bar{z}) = -\frac{1}{\pi N} \lim_{\kappa \to 0} \sum_{i=1}^{N} \partial_{\bar{z}} b_i(z, \bar{z}, \kappa). \qquad (17)$$

To deal with the partial derivative appearing in Eq. (17), we use Eq. (13) to formulate a similar set of consistency relations for the partial derivatives of the covariance matrices $\{\partial_{\bar{z}} \mathcal{C}_i^{(j)}\}$:

$$\partial_{\bar{z}} \mathcal{C}_i^{(j)} = -\mathcal{C}_i^{(j)} \left[\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} - F(\partial_{\bar{z}} \mathcal{C}_{\partial i \backslash j}^{(i)}) \right] \mathcal{C}_i^{(j)}. \qquad (18)$$

Similarly, the derivative of the "true" covariance matrix at i is given according to Eq. (15) by

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[2]In fact, this factorisation also applies in the fully connected case, though the cause is statistical rather than topological.

$$\partial_{\bar{z}}\mathcal{C}_i = -\mathcal{C}_i\left[\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} - F(\partial_{\bar{z}}\mathcal{C}_{\partial i}^{(i)})\right]\mathcal{C}_i. \tag{19}$$

Equations (13), (15), (18), and (19) comprise our main result. For a given treelike matrix A, one iterates Eqs. (13) and (18) together until convergence. The "true" marginals are then recovered via Eqs. (15) and (19), and finally the spectral density is given by Eq. (17).

FULLY CONNECTED LIMIT

To assess our approach from a theoretical viewpoint, we (re)derive the generalized Girko's law of Ref. [5] in the fully connected limit. To do so, we first rewrite $A^s \to v A^s$ with $v^2 = (1-\tau)/(1+\tau)$.

Consider statistically independent matrices A^h and A^s, with

$$\mathbb{E}(A_{ij}^h) = 0, \quad \mathbb{E}(|A_{ij}^h|^2) = (1+\tau)/2c, \tag{20}$$

and similarly for A^s. Here, τ is a parameter controlling the degree of hermiticity; at $\tau=1$, A is Hermitian, whereas, at $\tau=0$, A is maximally non-Hermitian. We will study the limit of large c, by which we understand $k_i \to c$ and $c \to \infty$. First notice that Eqs. (13) and (15), along with the correlations given above, imply $\mathcal{C}_i^{(j)} = \mathcal{C}_i + O(1/c)$. Moreover, upon introducing

$$\begin{pmatrix} a & i\bar{b} \\ ib & d \end{pmatrix} = \lim_{c\to\infty}\frac{1}{c}\sum_{l\in\partial i}\mathcal{C}_i, \tag{21}$$

the spectral density takes the following form:

$$\rho(z,\bar{z}) = -\frac{1}{\pi}\partial_{\bar{z}}b(z,\bar{z}). \tag{22}$$

Then, in the limit $c\to\infty$ Eq. (15) yields

$$\begin{pmatrix} a & i\bar{b} \\ ib & d \end{pmatrix} = \frac{1}{ad+|\tau b+z|^2}\begin{pmatrix} a & -i(\tau b+z) \\ -i(\tau\bar{b}+\bar{z}) & d \end{pmatrix}, \tag{23}$$

where we have set κ harmlessly to zero. This equation is easily solved giving

$$\rho(z,\bar{z}) = \begin{cases} \dfrac{1}{\pi(1-\tau^2)} & \text{for} \quad \left(\dfrac{x}{1+\tau}\right)^2 + \left(\dfrac{y}{1-\tau}\right)^2 \leq 1, \\ 0 & \text{else.} \end{cases} \tag{24}$$

This is the well known generalized Girko's law of Ref. [5].

NUMERICAL RESULTS

For ensembles of sparse random matrices, the cavity equations can be solved quickly by computer. We present the results, in comparison with direct diagonalization, for two cases: (i) symmetrically connected Poissonian random graphs with average connectivity c and with asymmetric Gaussian edge weights with zero mean and variance $1/c$ and
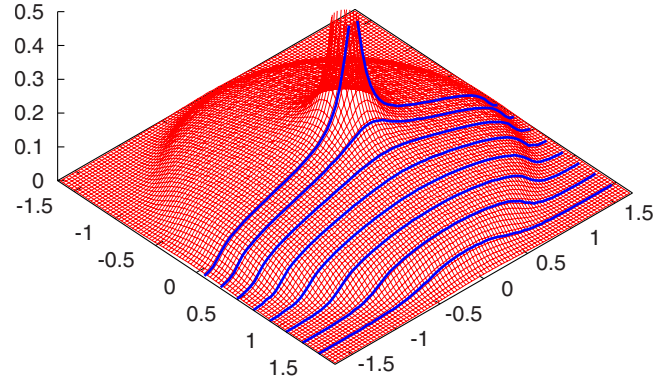

FIG. 1. (Color online) Spectral density of symmetric Poissonian graphs with asymmetric Gaussian edge weights and average connectivity $c=5$. The red grid is a histogram of the eigenvalues of 10^5 samples and the blue solid lines are the result of the cavity equations, averaged over 1000 samples.

(ii) asymmetrically connected Poissonian graphs with edge weights drawn uniformly from the circle of radius $1/\sqrt{c}$.

Since numerical diagonalization of large matrices is a computationally demanding task, we have chosen to study relatively "small" matrices of size $N=1000$, even though the cavity equations are capable of handling matrices many orders of magnitude larger. In each case, we have diagonalized numerically 10^5 such matrices, having an overall of 10^8 complex eigenvalues, so as to have smooth two-dimensional histrograms.

For a given matrix, we use the cavity equations as a belief propagation algorithm: we iterate Eqs. (13) and (18) together until convergence is reached and, then, compute the spectral density from Eqs. (15), (19), and (17). The results are averaged over 1000 samples.

The results from the cavity equations and comparison with numerical diagonalization are presented in Figs. 1 and 2 for cases (i) and (ii), respectively. To give a better view of the detail, Fig. 3 shows a pair of slices taken from Fig. 2. Notice that the ensembles in both cases satisfy the conditions for Girko's law in the limit $c \to \infty$. However, it is evident
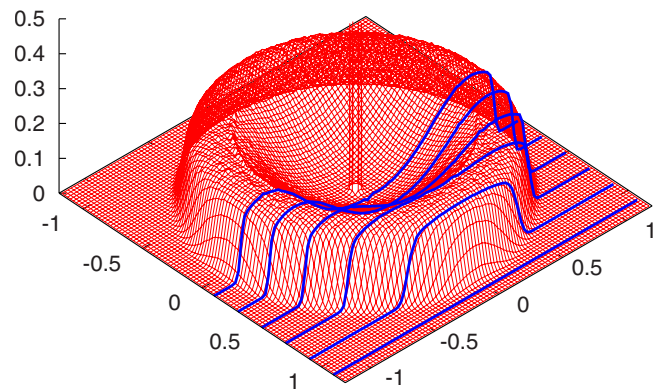

FIG. 2. (Color online) Spectral density of asymmetric Poissonian graphs with unitary edge weights and average connectivity $c=2$. The red grid is a histogram of the eigenvalues of 10^5 samples and the blue solid lines are the result of the cavity equations, averaged over 1000 samples.
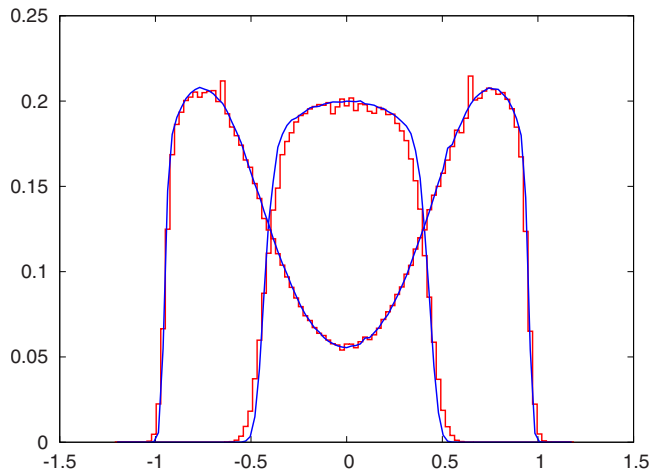
FIG. 3. (Color online) Cuts along the lines $x=0.3$ and $x=0.9$ from Fig. 2.

from the figures that, for finite c, they have spectral densities dramatically different both from each other and from the limiting case of Girko's law. Apart from small discrepancies near the boundaries due to the discretization the histogram introduces, the comparison shows excellent agreement.

CONCLUSIONS

In this paper we have considered the problem of determining the mean spectral density of ensembles of sparse non-Hermitian random matrices. Following standard steps [5,15], the problem can be phrased in terms of a Gaussian integral, which we interpret in the language of statistical mechanics of disordered systems as in Ref. [17]. Within this framework, and following the steps of Ref. [10], we apply the cavity method, deriving a set of equations whose solution characterizes the spectral density of a given sparse matrix.

As we have shown, our work puts analytical results such as the generalized Girko's law within easy reach, when appropriate limits are taken. Moreover, a numerical solution for finite-size matrices can be easily obtained by belief propagation, giving results in excellent agreement with those of direct diagonalization.

In the case of dense matrices (both Hermitian and non-Hermitian), past studies using the techniques of supersymmetry and replica analysis have found considerable success. However, applied to sparse matrices, these approaches have not proved as fruitful, leading to a set of saddle-point equations which have resisted numerical solution for over 17 years.

To make contact with these other approaches, note that, in the ensemble average, the cavity and replica methods are known to be equivalent. In fact, the solution we have given here is common to all approaches and can also be derived through a careful treatment of the aforementioned saddle-point equations [18].

For more than 50 years, rigorous analysis of the circular law has proven to be rather difficult [4]. Therefore, it would be a very exciting prospect to reconsider this problem by using rigorous techniques introduced in the area of spin glasses (e.g., the interpolation method). Work along these lines is under way [19].

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